MATH 105A and 110A Review: Gram-Schmidt process

1. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

Is $\mathcal{B}$ orthogonal or orthonormal?

Solution: The set is not orthognal because

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \neq 0.$$

Since $\mathcal{B}$ is not orthogonal, then it is not orthonormal.

2. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Is $\mathcal{B}$ orthogonal or orthonormal?

Solution: The set is orthognal because

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0, \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0,$$

But $\mathcal{B}$ is not orthonormal because

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \neq 1.$$

3. The basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right\}$$

is a basis for a plane in $\mathbb{R}^3$. Find an orthonormal basis for the same plane.

Solution: Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}.$$

Then,

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

and so,

$$w_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

Then,

$$u_2 = v_2 - \operatorname{proj}_{u_1} v_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} - \frac{2+2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

Hence,

$$w_2 = \frac{1}{3} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Our orthonomral basis is

$$\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$